Currents and Potentials along Leaky Ground-Return Conductors *

By E. D. SUNDE

THE problem of current and potential propagation for long conductors having large leakance to ground arises in connection with certain railway electrification problems, such as inductive effects in exposed communication lines, and voltages to ground of the tracks and of nearby underground cables. Impressed voltages in exposed lines are due partly to induction and partly to earth potential differences, and the latter in particular depend to a considerable extent on the mode of propagation of the track current. Voltages to ground of underground telephone cables depend on the mode of propagation along the cables of current produced in these by nearby railway electrification. These voltages may under certain conditions raise questions as to the possibility of hazard or, in case of direct current, give rise to electrolytic effects.

In considering propagation along earth-return conductors of the above kind, it is necessary to include certain effects which can be neglected in circuits having small leakance. One of the quantities involved in the differential equation for the current at a point of a ground-return conductor is the axial electric force produced in the ground adjacent to the conductor. For a given frequency and earth resistivity, this electric force at the point under consideration depends partly on the axial current distribution and partly on the leakage current distribution along the entire length of the conductor. The component depending on the axial current distribution is the vector potential multiplied by $-i\omega$, ω being the radian frequency, while the other component is the negative gradient of the earth potential, which depends on the leakage current distribution along the conductor. In the customary treatment, applying to conductors of small leakance, the axial current is assumed practically constant for great distances along the conductor, so that the first component becomes the negative product of axial conductor current at the point under consideration and the external earth-return impedance of the conductor. Furthermore, the earth potential is neglected or assumed constant along the conductor, so that the second component vanishes.

* Digest of a paper to be presented at the Winter Convention of the American Institute of Electrical Engineers, New York, January 25–29, 1937, and published in full in *Electrical Engineering*, Vol. 55, No. 12, pp. 1338–1346, December, 1936.

For conductors with large leakance, these simplifying assumptions are not justified. When the earth resistivity is uniform or varies with depth only, the electric force may be formulated in terms of the current along the entire length of the conductor, in which case the usual differential equation for the current is replaced by an integro-differential equation. A general solution of this equation and for conductor and earth potentials has been obtained. For homogeneous earth, rigorous as well as approximate solutions of special cases of interest in connection with the railway electrification problems mentioned above have also been derived.

One of these cases is of general interest since it may be regarded as fundamental to the solution of the general case of an arbitrary impressed electric force along the conductor. In this case a voltage $2V(0)$ is impressed across a break in the conductor at a certain point, which may be taken as the origin. The conductor current and potential are given by rather complicated integrals, which, in order to obtain practical formulas, may be expanded in series as:

$$I(x) = I_1(x) + I_2(x); \qquad I_2(x) = I_{21}(x) + I_{22}(x), \tag{1}$$

$$V(x) = V_1(x) + V_2(x); \qquad V_2(x) = V_{21}(x) + V_{22}(x), \tag{2}$$

where $I_{22}(x)$ and $V_{22}(x)$ may again be written as the sum of two terms, etc. The first terms in these expansions are:

$$I_1(x) = I_1(-x) = V(0)\frac{G(\Gamma)}{\Gamma}e^{-\Gamma x} = I(0)e^{-\Gamma x}, \tag{3}$$

$$V_1(x) = -V_1(-x) = V(0)e^{-\Gamma x} = I(0)\frac{\Gamma}{G(\Gamma)}e^{-\Gamma x}, \tag{4}$$

where $x \geq 0$ and:

$$\Gamma = \sqrt{Z(\Gamma)G(\Gamma)}.$$

$$Z(\Gamma) = z + \frac{i\omega\nu}{2\pi}\log_e\frac{1.85\cdots}{a\alpha}.$$

$$G(\Gamma) = \left[g^{-1} + \frac{\rho}{\pi}\log_e\frac{1.12\cdots}{a\Gamma}\right]^{-1}.$$

$$\alpha = (i\omega\nu\rho^{-1} + \Gamma^2)^{1/2}.$$

$$\omega = 2\pi f.$$

f = frequency in cycles per second.

x = distance from origin, in meters.

a = conductor radius, in meters.

z = internal impedance in ohms per meter.

g = leakage conductance in ohms per meter.

ρ = earth resistivity in meter-ohms.

$\nu = 4\pi\cdot 10^{-7}$ henries per meter.

The transcendental equation defining the propagation constant Γ may be solved by successive approximations; a convenient first approximation is $\Gamma = \Gamma_1 = \sqrt{gZ(0)}$, $Z(0)$ being the earth-return self-impedance of the conductor. Equations (3) and (4) are of the same form as the solution for conductors of small leakance, except that the propagation constant for the latter is taken equal to Γ_1 above. The effect of the earth potential appears in a first approximation as the second term in the expression for $G(\Gamma)$.

For earth resistivities within the usual range and for electric railway tracks or underground cables the two terms in the expression for $G(\Gamma)$ are frequently of the same order of magnitude. Appreciable errors may therefore be obtained by neglecting the second term, and in correlating the results of measurements this must be kept in mind.

The second terms in the expansions are given below, but may be neglected in the range of most practical applications.

$$I_{21}(x) = I_{21}(-x) = -I(0)\frac{G(\Gamma)\rho}{4\pi}\{(1+\Gamma x)e^{\Gamma x}Ei(\Gamma x) - (1-\Gamma x)e^{-\Gamma x}[Ei(-\Gamma x)+i\pi]\}, \quad (5)$$

$$V_{21}(x) = -V_{21}(-x) = I(0)\frac{\Gamma\rho}{4\pi}\{\Gamma x e^{\Gamma x}Ei(\Gamma x) - \Gamma x e^{-\Gamma x}[Ei(-\Gamma x)+i\pi] - 2\}, \quad (6)$$

where $Ei(u) = \int_u^\infty \frac{e^{-u}}{u}du$ is the exponential integral.

For sufficiently large values of Γx the bracket terms of expressions (5) and (6) vanish as $-8/(\Gamma x)^3$ and $4/(\Gamma x)^2$, respectively, so that in this case the second terms in the expansions predominate.